

LINEAR STATISTICS OF LOW-LYING ZEROS OF L -FUNCTIONS

by C. P. HUGHES[†] and Z. RUDNICK[‡]

(Raymond and Beverly Sackler School of Mathematical Sciences, Tel Aviv University, Tel Aviv
69978, Israel)

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Abstract

We consider linear statistics of the scaled zeros of Dirichlet L -functions, and show that the first few moments converge to the Gaussian moments. The number of Gaussian moments depends on the particular statistic considered. The same phenomenon is found in random matrix theory, where we consider linear statistics of scaled eigenphases for matrices in the unitary group. In that case the higher moments are no longer Gaussian. We conjecture that this also happens for Dirichlet L -functions.

1. Introduction

Let q be an odd prime and χ a Dirichlet character modulo q . For $\Re(s) > 1$ the Dirichlet L -function $L(s, \chi)$ is defined as

$$L(s, \chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.$$

Each function $L(s, \chi)$ has an infinite set of non-trivial zeros $\frac{1}{2} + i\gamma_{\chi, j}$ which can be ordered so that

$$\dots \leq \Re(\gamma_{\chi, -2}) \leq \Re(\gamma_{\chi, -1}) < 0 \leq \Re(\gamma_{\chi, 1}) \leq \Re(\gamma_{\chi, 2}) \leq \dots$$

Note that we do not assume the generalized Riemann hypothesis (GRH) since we allow the $\gamma_{\chi, j}$ to be complex.

Denote by $N(T, \chi)$ number of non-trivial zeros such that $0 < \Re(\gamma_{\chi, j}) < T$. Then for fixed $T > 0$,

$$\frac{1}{q-2} \sum_{\chi \neq \chi_0} N(T, \chi) \sim \frac{T}{2\pi} \log \frac{qT}{2\pi e} \quad \text{as } q \rightarrow \infty,$$

where the sum is taken over all the $q-2$ non-trivial characters modulo the prime q ; see Titchmarsh [22], Siegel [19], Selberg [18]. We will therefore scale the zeros by defining

$$x_{\chi, j} := \frac{\log q}{2\pi} \gamma_{\chi, j}.$$

[†] Current address: American Institute of Mathematics, 360 Portage Ave, Palo Alto, CA 94306-2244, USA.

E-mail: hughes@aimath.org

[‡] E-mail: rudnick@post.tau.ac.il

The purpose of this paper is to consider linear statistics of the low-lying $x_{\chi,j}$. Let f be a rapidly decaying even test function, and consider the linear statistic

$$W_f(\chi) := \sum_{j=-\infty}^{\infty} f(x_{\chi,j}).$$

Linear statistics for low-lying zeros of several families of L -functions were investigated systematically by Katz and Sarnak [10] and by Iwaniec, Luo and Sarnak [9] where they were called ‘one-level densities’. We prefer to use the terminology ‘linear statistic’ which is traditional in random matrix theory.

Define the χ -average of $W_f(\chi)$ as

$$\langle W_f \rangle_q := \frac{1}{q-2} \sum_{\chi \neq \chi_0} W_f(\chi).$$

In order to understand the distribution of $W_f(\chi)$ we calculate its first few moments $\langle W_f^m \rangle_q$. In section 3 we prove that if $\text{supp } \widehat{f} \subseteq [-2, 2]$, then the mean of $W_f(\chi)$ is $\int_{-\infty}^{\infty} f(x) dx$, and in section 4 we show that the variance converges to $\int_{-1}^1 |u| \widehat{f}(u)^2 du$ if $\text{supp } \widehat{f} \subseteq [-1, 1]$. In section 5 we show that if $\text{supp } \widehat{f} \subset (-2/m, 2/m)$ (for $m \geq 2$) then the first m moments of W_f converge to the first m moments of a normal random variable with mean and variance as above.

If all moments of $W_f(\chi)$ were Gaussian, then we would be able to conclude that W_f was normally distributed. Indeed, it follows from the work of Selberg [18] that scaling the $\gamma_{\chi,j}$ by anything much less than $\log q$ leads to a Gaussian distribution. However, with scaling on the order of $\log q$, this cannot be the case for all f since taking f to be an indicator function, the limiting distribution is discrete. We therefore say that W_f displays *mock-Gaussian* behaviour.

As suggested by Katz and Sarnak [10], one may try to model properties of $W_f(\chi)$ by random matrix theory. Let U be an $N \times N$ unitary matrix, with eigenvalues $e^{i\theta_n}$. The statistical distribution of $(N/2\pi)\theta_n$ has been conjectured to converge to the empirical distribution of $x_{\chi,j}$ as q and N both tend to infinity

Therefore, as an aide to understanding $W_f(\chi) = \sum_{j=-\infty}^{\infty} f(x_{\chi,j})$, one might wish to calculate the moments of $\sum_{n=1}^N f((N/2\pi)\theta_n)$. However, since the θ_n are angles, it is more natural (and indeed more convenient) to consider the 2π -periodic function

$$F_N(\theta) := \sum_{j=-\infty}^{\infty} f\left(\frac{N}{2\pi}(\theta + 2\pi j)\right)$$

and model $W_f(\chi)$ by

$$Z_f(U) := \sum_{j=1}^N F_N(\theta_j),$$

where U is an $N \times N$ unitary matrix with eigenangles $\theta_1, \dots, \theta_N$. Note that the scaling $N/2\pi$ (the mean density) is equivalent to the scaling $\log q/2\pi$ for the zeros of L -functions.

Our results for $Z_f(U)$ are given in sections 6,7. Write \mathbb{E} to denote the average over the unitary

group with Haar measure; then without any restrictions on the support of the function f , we prove that $\mathbb{E}\{Z_f(U)\} = \int_{-\infty}^{\infty} f(x) dx$, and that the variance tends to

$$\sigma(f)^2 = \int_{-\infty}^{\infty} \min(1, |u|) \widehat{f}(u)^2 du. \tag{1.1}$$

Observe that this is in complete agreement with the mean and variance of $W_f(\chi)$ if \widehat{f} has the same support restrictions. Furthermore, we show in section 6.1 that for any integer $m \geq 2$, if $\text{supp } \widehat{f} \subseteq [-2/m, 2/m]$, then

$$\lim_{N \rightarrow \infty} \mathbb{E} \{ (Z_f - \mathbb{E}\{Z_f\})^m \} = \begin{cases} 0 & \text{if } m \text{ odd,} \\ \frac{m!}{2^{m/2}(m/2)!} \sigma^m & \text{if } m \text{ even,} \end{cases}$$

where σ^2 , the variance, is given in (1.1). These are the moments of a normal random variable, so again we see mock-Gaussian behaviour, with the same restrictions on the support of \widehat{f} as in $W_f(\chi)$.

To understand the mock-Gaussian behaviour, note that if we had defined

$$F_N^{(L)}(\theta) = \sum_{j=-\infty}^{\infty} f(L(\theta + 2\pi j)),$$

where $L \rightarrow \infty$ subject to $L/N \rightarrow 0$, then Soshnikov [21] (see also [4]) has shown that the mean of $Z_f^{(L)}(U)$ converges to $(N/2\pi L) \int_{-\infty}^{\infty} f(x) dx$, and that the centred random variable $Z_f^{(L)} - \mathbb{E}\{Z_f^{(L)}\}$ converges in distribution to a normal random variable with mean zero and variance $\int_{-\infty}^{\infty} \widehat{f}(u)^2 |u| du$. Our scaling is $L = N/2\pi$, which is just outside the range of Soshnikov’s result. Indeed, note that the variance (1.1) is different if $\text{supp } \widehat{f} \not\subseteq [-1, 1]$.

In section 7 we show that all moments of Z_f can be calculated exactly within random matrix theory, without any restrictions on the support of \widehat{f} . They are given by a complicated expression, but are certainly not Gaussian moments in general. The moments of $Z_f(U)$ grow sufficiently slowly that they uniquely determine its distribution.

Finally, in section 8 we apply the results of sections 3,4 to show that, under the assumption of GRH, for each q , there exist χ such that the height of the lowest zero is less than a quarter of the expected height. We also obtain a similar result where a positive proportion of L -functions have their first zero less than 0.633 times the expected height.

Linear statistics of the high zeros of a fixed L -function also show mock-Gaussian behaviour [7], having the same moments as the linear statistics of low-lying zeros considered in this paper.

Moments of linear statistics in other classical compact groups, like $\text{SO}(2N)$, $\text{SO}(2N + 1)$ and $\text{Sp}(2N)$ also show mock-Gaussian behaviour [8]. Other L -functions can be modelled by these groups. For example, in the case of quadratic L -functions mock-Gaussian behaviour can be deduced from the work of Rubinstein [17]. Specifically, if $\text{supp } \widehat{f} \subset (-1/m, 1/m)$ then the first m moments are Gaussian with mean $\widehat{f}(0) - \int_0^1 \widehat{f}(u) du$ and variance $4 \int_0^{1/2} u \widehat{f}(u)^2 du$, exactly as in the group $\text{Sp}(2N)$. We remark that this is half the unitary range. Assuming GRH, the results of Özlück and Snyder [15] show that the mean is indeed $\widehat{f}(0) - \int_0^1 \widehat{f}(u) du$ so long as $\text{supp } \widehat{f} \subset (-2, 2)$, in the sense that

$$\frac{1}{D} \sum_d e^{-\pi d^2/D^2} \sum_{\gamma} f\left(\frac{\gamma \log D}{2\pi}\right) = \widehat{f}(0) - \int_0^1 \widehat{f}(u) du + o(1).$$

We note that the arguments given here to study moments of the linear statistic show that the ‘ n -level densities’ [10, 17] of this family of L -functions coincide with those of the unitary group, in a suitable range, by purely combinatorial arguments. Since that is not our purpose here we leave it for the reader.

Throughout all this paper, the Fourier transform is $\widehat{f}(u) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x u} dx$, and thus the inverse transform is $f(x) = \int_{-\infty}^{\infty} \widehat{f}(u)e^{2\pi i x u} du$.

2. The scaled level density W_f

2.1. Zeros and the explicit formula

A Dirichlet character $\chi : \mathbb{N} \rightarrow \mathbb{C}$ is a function such that $\chi(n + q) = \chi(n)$ for all n ; $\chi(n) = 0$ if n and q have a common divisor; and $\chi(mn) = \chi(m)\chi(n)$ for all m, n . We say χ is the trivial character modulo q (denoted by χ_0) if $\chi(n) = 1$ for all n coprime to q . We say that χ is even if $\chi(-1) = 1$, and odd if $\chi(-1) = -1$.

The explicit formula is the following relation between a sum over zeros of $L(s, \chi)$ and a sum over prime powers. To describe it, let

$$a(\chi) = \begin{cases} 0, & \chi \text{ even,} \\ 1, & \chi \text{ odd,} \end{cases}$$

and let $h(r)$ be any even analytic function in the strip $-c \leq \Im(r) \leq 1 + c$ (for $c > 0$) such that $|h(r)| \leq A(1 + |r|)^{-(1+\delta)}$ (for $r \in \mathbb{R}$, $A > 0$, $\delta > 0$). Set $g(u) = (1/2\pi) \int_{-\infty}^{\infty} h(r)e^{-iru} dr$, so that $h(r) = \int_{-\infty}^{\infty} g(u)e^{iru} du$. Then

$$\begin{aligned} \sum_j h(\gamma_{j,\chi}) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} h(r) (\log q + G_\chi(r)) dr \\ &\quad - \sum_n \frac{\Lambda(n)}{\sqrt{n}} g(\log n) (\chi(n) + \bar{\chi}(n)), \end{aligned} \tag{2.1}$$

where

$$G_\chi(r) = \frac{\Gamma'}{\Gamma}(\frac{1}{2} + a(\chi) + ir) + \frac{\Gamma'}{\Gamma}(\frac{1}{2} + a(\chi) - ir) - \frac{1}{2} \log \pi$$

and the von Mangoldt function $\Lambda(n)$ is defined as $\log p$ if $n = p^k$ is a prime power, and zero otherwise.

2.2. A decomposition of W_f

For test functions f define the scaled level density $W_f(\chi)$ as

$$W_f(\chi) := \sum_j f\left(\frac{\log q}{2\pi} \gamma_{\chi,j}\right),$$

the sum over all non-trivial zeros of $L(s, \chi)$.

DEFINITION 2.1 $f(x)$ is an admissible test function for $W_f(\chi)$ if it is a real, even function, whose Fourier transform $\widehat{f}(u)$ is compactly supported, and such that $f(r) \ll (1 + |r|)^{-1-\delta}$.

We will transform W_f into a sum over prime powers by using the explicit formula for $L(s, \chi)$. In (2.1), take $h(r) = f\left(\frac{\log q}{2\pi}r\right)$, so that $g(u) = \frac{1}{\log q} \widehat{f}\left(\frac{u}{\log q}\right)$, and note that the conditions on \widehat{f} easily imply the analyticity and decay condition on $h(r)$ in the explicit formula. We then get a decomposition of $W_f(\chi)$ as

$$W_f(\chi) = \overline{W_f}(\chi) + W_f^{\text{osc}}(\chi), \tag{2.2}$$

where

$$\overline{W_f}(\chi) := \frac{1}{2\pi} \int_{-\infty}^{\infty} f\left(\frac{\log q}{2\pi}r\right) (\log q + G_\chi(r)) \, dr$$

and (an oscillatory term)

$$W_f^{\text{osc}}(\chi) := -\frac{1}{\log q} \sum_n \frac{\Lambda(n)}{\sqrt{n}} \widehat{f}\left(\frac{\log n}{\log q}\right) (\chi(n) + \bar{\chi}(n)). \tag{2.3}$$

The first term $\overline{W_f}(\chi)$ gives

$$\overline{W_f}(\chi) = \int_{-\infty}^{\infty} f(x) \, dx + \mathcal{O}\left(\frac{1}{\log q}\right),$$

which is asymptotically independent of χ .

3. The expectation of W_f

The ‘expectation’ of W_f is defined as the average over all $q - 2$ non-trivial characters modulo the prime q

$$\langle W_f \rangle_q := \frac{1}{q - 2} \sum_{\chi \neq \chi_0} W_f(\chi).$$

THEOREM 3.1 *Let f be an admissible function, and assume $\text{supp}(\widehat{f}) \subseteq [-2, 2]$. Then, as $q \rightarrow \infty$,*

$$\langle W_f \rangle_q = \int_{-\infty}^{\infty} f(x) \, dx + \mathcal{O}\left(\frac{1}{\log q}\right).$$

Proof. We will use the decomposition (2.2) and average over χ :

$$\langle W_f \rangle_q = \langle \overline{W_f} \rangle_q + \langle W_f^{\text{osc}} \rangle_q.$$

Since $\overline{W_f}$ is asymptotically constant for any f , we have

$$\langle \overline{W_f} \rangle_q = \int_{-\infty}^{\infty} f(x) \, dx + \mathcal{O}\left(\frac{1}{\log q}\right)$$

and thus it will suffice to show that

$$\langle W_f^{\text{osc}} \rangle_q = \mathcal{O}\left(\frac{1}{\log q}\right).$$

We will show this under the assumption $\text{supp}(\widehat{f}) \subseteq [-2, 2]$.

We have by (2.3) that

$$\left\langle W_f^{\text{osc}} \right\rangle_q = -\frac{1}{\log q} \sum_n \frac{\Lambda(n)}{\sqrt{n}} \widehat{f}\left(\frac{\log n}{\log q}\right) (\langle \chi(n) \rangle_q + \langle \bar{\chi}(n) \rangle_q).$$

The mean value of χ is

$$\langle \chi(n) \rangle_q = \langle \bar{\chi}(n) \rangle_q = \begin{cases} 1, & n \equiv 1 \pmod{q}, \\ 0, & q \mid n, \\ -\frac{1}{q-2}, & n \not\equiv 0, 1 \pmod{q}. \end{cases}$$

Thus we find

$$\left\langle W_f^{\text{osc}} \right\rangle_q = \frac{-2}{\log q} \sum_{n \equiv 1 \pmod{q}} \frac{\Lambda(n)}{\sqrt{n}} \widehat{f}\left(\frac{\log n}{\log q}\right) + \frac{2}{\log q} \frac{1}{q-2} \sum_{n \not\equiv 1, 0 \pmod{q}} \frac{\Lambda(n)}{\sqrt{n}} \widehat{f}\left(\frac{\log n}{\log q}\right).$$

Assume that $\text{supp}(\widehat{f}) \subseteq [-\alpha, \alpha]$ for $\alpha > 0$. Then the sum is over $n \leq q^\alpha$. Since \widehat{f} is bounded, we may replace it by 1 over that range, and we therefore have

$$\left\langle W_f^{\text{osc}} \right\rangle_q \ll \frac{1}{\log q} \sum_{\substack{n \equiv 1 \pmod{q} \\ n \leq q^\alpha}} \frac{\Lambda(n)}{\sqrt{n}} + \frac{1}{\log q} \frac{1}{q-2} \sum_{n \leq q^\alpha} \frac{\Lambda(n)}{\sqrt{n}}. \tag{3.1}$$

To deal with the first sum in (3.1) one could replace primes by integers upon noting that $\Lambda(n) \ll \log n$, obtaining

$$\begin{aligned} \frac{1}{\log q} \sum_{\substack{n \equiv 1 \pmod{q} \\ n \leq q^\alpha}} \frac{\Lambda(n)}{\sqrt{n}} &\ll \frac{1}{\log q} \sum_{\substack{n \equiv 1 \pmod{q} \\ n \leq q^\alpha}} \frac{\log n}{\sqrt{n}} \\ &= \frac{1}{\log q} \sum_{m < q^{\alpha-1}} \frac{\log(mq+1)}{\sqrt{mq+1}} \\ &\ll \frac{1}{\sqrt{q}} q^{(\alpha-1)/2} \end{aligned}$$

which vanishes for $\alpha < 2$. However, one gets a slightly stronger result by the Brun–Titchmarsh theorem [13], which says that if $\pi(x; q, a)$ is the number of primes $p \leq x$, $p \equiv a \pmod{q}$, where q and a are coprime, then for $x > 2q$,

$$\pi(x; q, a) < \frac{2x}{\varphi(q) \log(x/q)}.$$

Therefore, for $\alpha > 1$,

$$\begin{aligned} \frac{1}{\log q} \sum_{\substack{n \equiv 1 \pmod{q} \\ n \leq q^\alpha}} \frac{\Lambda(n)}{\sqrt{n}} &\ll \frac{1}{\log q} \sum_{\substack{p \equiv 1 \pmod{q} \\ p \leq q^\alpha}} \frac{\log p}{\sqrt{p}} \\ &\ll \frac{1}{\log q} \int_{2q}^{q^\alpha} \frac{\log x}{\sqrt{x}} \frac{1}{q \log(x/q)} dx \ll \frac{1}{\log q} q^{-1+\alpha/2} \end{aligned}$$

which vanishes for $\alpha \leq 2$.

To deal with the second sum in (3.1), one could similarly replace primes by integers, and note that

$$\frac{1}{\log q} \frac{1}{q-2} \sum_{n \leq q^\alpha} \frac{\log n}{\sqrt{n}} \ll q^{-1+\alpha/2}$$

which vanishes if $\alpha < 2$. Again this result can be strengthened by using the prime number theorem, since

$$\begin{aligned} \frac{1}{\log q} \frac{1}{q-2} \sum_{\substack{n \leq q^\alpha \\ n \neq 0,1 \pmod q}} \frac{\Lambda(n)}{\sqrt{n}} &\ll \frac{1}{\log q} \frac{1}{q} \sum_{p \leq q^\alpha} \frac{\log p}{\sqrt{p}} \\ &\ll \frac{1}{\log q} \frac{1}{q} \int_2^{q^\alpha} \frac{\log x}{\sqrt{x}} \frac{dx}{\log x} \ll \frac{1}{\log q} q^{-1+\alpha/2} \end{aligned}$$

which vanishes for $\alpha \leq 2$.

Thus $\langle W_f^{\text{osc}} \rangle_q \ll \frac{q^{-1+\alpha/2}}{\log q}$ and so if $\alpha \leq 2$ we find $\langle W_f^{\text{osc}} \rangle_q \rightarrow 0$ as required.

REMARK 1 Set

$$W_f^{(t)}(\chi) = \sum_{\gamma_{\chi,j}} f\left(\frac{\log q}{2\pi}(\gamma_{\chi,j} - t)\right)$$

which is like $W_f(\chi)$ but with the zeros shifted by height t . Averaging over all characters modulo q , and doing an extra smooth average over t , the expected value of $W_f^{(t)}$ converges to $\int_{-\infty}^{\infty} f(x) dx$ without any restriction on the support of \widehat{f} .

4. The variance of W_f

THEOREM 4.1 *Let f be an admissible function and assume $\text{supp } \widehat{f} \subseteq [-1, 1]$; then the variance of W_f tends to*

$$\sigma(f)^2 = \int_{-1}^1 |u| \widehat{f}(u)^2 du. \tag{4.1}$$

Proof. The variance of W_f is, by Theorem 3.1,

$$\left\langle (W_f - \langle W_f \rangle_q)^2 \right\rangle_q = \left\langle (W_f^{\text{osc}})^2 \right\rangle_q + \mathcal{O}\left(\frac{1}{\log q}\right),$$

and by (2.3),

$$\begin{aligned} \left\langle (W_f^{\text{osc}})^2 \right\rangle_q &= \frac{1}{(\log q)^2} \sum_{n_1} \sum_{n_2} \frac{\Lambda(n_1)}{\sqrt{n_1}} \frac{\Lambda(n_2)}{\sqrt{n_2}} \widehat{f}\left(\frac{\log n_1}{\log q}\right) \widehat{f}\left(\frac{\log n_2}{\log q}\right) \\ &\quad \times (\langle \chi(n_1)\chi(n_2) \rangle_q + \langle \chi(n_1)\bar{\chi}(n_2) \rangle_q + \langle \bar{\chi}(n_1)\chi(n_2) \rangle_q + \langle \bar{\chi}(n_1)\bar{\chi}(n_2) \rangle_q). \end{aligned}$$

Now

$$\langle \chi(n_1)\chi(n_2) \rangle_q = \langle \chi(n_1n_2) \rangle_q = \begin{cases} 1, & n_1n_2 \equiv 1 \pmod q, \\ 0, & n_1 \text{ or } n_2 \equiv 0 \pmod q, \\ \frac{-1}{q-2}, & \text{otherwise} \end{cases}$$

and

$$\langle \chi(n_1)\bar{\chi}(n_2) \rangle_q = \begin{cases} 1, & n_1 \equiv n_2 \not\equiv 0 \pmod{q}, \\ 0, & n_1 \text{ or } n_2 \equiv 0 \pmod{q}, \\ \frac{-1}{q-2}, & \text{otherwise.} \end{cases}$$

Since we assume $\text{supp } \hat{f} \subseteq [-1, 1]$, we need only consider $n_1, n_2 \leq q$. Therefore, writing \bar{n}_1 for the inverse of n_1 modulo q ,

$$\begin{aligned} \langle (W_f^{\text{osc}})^2 \rangle_q &= \frac{1}{(\log q)^2} \left(2 \sum_{n_1=2}^{q-1} \frac{\Lambda(n_1)^2}{n_1} \hat{f}\left(\frac{\log n_1}{\log q}\right)^2 \right. \\ &\quad \left. + 2 \sum_{n_1=2}^{q-1} \frac{\Lambda(n_1)}{\sqrt{n_1}} \frac{\Lambda(\bar{n}_1)}{\sqrt{\bar{n}_1}} \hat{f}\left(\frac{\log n_1}{\log q}\right) \hat{f}\left(\frac{\log \bar{n}_1}{\log q}\right) \right) \\ &\quad + \mathcal{O} \left(\frac{1}{q-2} \left(\frac{1}{\log q} \sum_{n \leq q} \frac{\Lambda(n)}{\sqrt{n}} \right)^2 \right). \end{aligned} \tag{4.2}$$

By the prime number theorem,

$$\frac{1}{\log q} \sum_{n \leq q} \frac{\Lambda(n)}{\sqrt{n}} \ll \frac{\sqrt{q}}{\log q}$$

and so the \mathcal{O} term in (4.2) is bounded by $1/(\log q)^2$.

Now,

$$\begin{aligned} \frac{2}{(\log q)^2} \sum_{n_1=2}^{q-1} \frac{\Lambda(n_1)}{\sqrt{n_1}} \frac{\Lambda(\bar{n}_1)}{\sqrt{\bar{n}_1}} \hat{f}\left(\frac{\log n_1}{\log q}\right) \hat{f}\left(\frac{\log \bar{n}_1}{\log q}\right) &\ll \sum_{\substack{p < q \\ p, \bar{p} \text{ prime}}} \frac{1}{\sqrt{p}} \frac{1}{\sqrt{\bar{p}}} \\ &= \sum_{\substack{p < q \\ p, \bar{p} \text{ prime}}} \frac{1}{\sqrt{k_p q + 1}}, \end{aligned}$$

where k_p is defined so that $p\bar{p} = 1 + k_p q$. By unique factorization, there are exactly two primes less than q which produce a given k_p , namely p and \bar{p} . Therefore there can be at most $\frac{1}{2}\pi(q)$ different k_p , all lying between 1 and $q - 1$, and so

$$\frac{1}{(\log q)^2} \sum_{n_1=2}^{q-1} \frac{\Lambda(n_1)}{\sqrt{n_1}} \frac{\Lambda(\bar{n}_1)}{\sqrt{\bar{n}_1}} \ll \sum_{k=1}^{\pi(q)/2} \frac{1}{\sqrt{kq + 1}} \ll \frac{1}{\sqrt{q}} \sqrt{\frac{q}{\log q}}.$$

Inserting this into (4.2) we have

$$\begin{aligned} \langle (W_f^{\text{osc}})^2 \rangle_q &= \frac{2}{(\log q)^2} \sum_{n=2}^{q-1} \frac{\Lambda(n)^2}{n_1} \widehat{f}\left(\frac{\log n}{\log q}\right)^2 + \mathcal{O}\left(\frac{1}{\sqrt{\log q}}\right) \\ &= \frac{2}{(\log q)^2} \sum_{p < q} \frac{(\log p)^2}{p} \widehat{f}\left(\frac{\log p}{\log q}\right)^2 + \mathcal{O}\left(\frac{1}{\sqrt{\log q}}\right) \\ &= \frac{2}{(\log q)^2} \int_2^q \frac{(\log x)^2}{x} \widehat{f}\left(\frac{\log x}{\log q}\right)^2 \frac{dx}{\log x} + \mathcal{O}\left(\frac{1}{\sqrt{\log q}}\right) \\ &= 2 \int_{\log 2 / \log q}^1 u \widehat{f}(u)^2 du + \mathcal{O}\left(\frac{1}{\sqrt{\log q}}\right) \end{aligned}$$

and so we see that, since $\widehat{f}(u)$ is an even function,

$$\lim_{q \rightarrow \infty} \langle (W_f^{\text{osc}})^2 \rangle_q = \int_{-1}^1 |u| \widehat{f}(u)^2 du$$

as required.

REMARK 2 Assuming GRH, Özlück [14] shows that the variance of linear statistics of the scaled zeros shifted by t and weighted by a smooth function K , converge to the weighted form of (4.1) so long as $\text{supp } \widehat{f} \subset (-2, 2)$, when averaged over t and over all characters of modulus less than q . In fact, random matrix theory suggests that (4.1) is the correct variance for all admissible functions (Theorem 6.2).

5. The moments of W_f

We now attempt to understand the *distribution* of the scaled level density W_f around its expected value. We will find that the first few moments of W_f converge to those of a Gaussian random variable with mean $\lim \overline{W}_f = \int_{-\infty}^{\infty} f(x) dx$ and variance

$$\int_{-\infty}^{\infty} \min(1, |u|) \widehat{f}(u)^2 du .$$

THEOREM 5.1 *Let f be an admissible function, and assume that*

$$\text{supp } \widehat{f} \subseteq [-\alpha, \alpha], \quad \alpha > 0 . \tag{5.1}$$

If $m < 2/\alpha$, then the m th moment of W_f^{osc} is

$$\lim_{q \rightarrow \infty} \langle (W_f^{\text{osc}})^m \rangle_q = \begin{cases} \frac{m!}{2^{m/2}(m/2)!} \sigma(f)^m, & m \text{ even,} \\ 0, & m \text{ odd,} \end{cases}$$

where $\sigma(f)^2$, the variance, is given in (4.1).

Proof. By (2.3), we have

$$W_f^{\text{osc}}(\chi) = -\frac{1}{\log q} \sum_n \frac{\Lambda(n)}{\sqrt{n}} \widehat{f}\left(\frac{\log n}{\log q}\right) (\chi(n) + \bar{\chi}(n)).$$

This gives

$$\begin{aligned} (W_f^{\text{osc}}(\chi))^m &= \left(-\frac{1}{\log q}\right)^m \prod_{j=1}^m \sum_{n_j} \frac{\Lambda(n_j)}{\sqrt{n_j}} \widehat{f}\left(\frac{\log n_j}{\log q}\right) (\chi(n_j) + \bar{\chi}(n_j)) \\ &= \sum_{S \subseteq \{1, \dots, m\}} J(S)(\chi), \end{aligned}$$

where for any subset of indices $S \subseteq \{1, \dots, m\}$, the summand $J(S)(\chi)$ corresponds to the different ways of picking χ and $\bar{\chi}$:

$$J(S)(\chi) := \left(-\frac{1}{\log q}\right)^m \sum_{n_1, \dots, n_m} \prod_{j=1}^m \frac{\Lambda(n_j)}{\sqrt{n_j}} \widehat{f}\left(\frac{\log n_j}{\log q}\right) \chi\left(\prod_{j \in S} n_j\right) \bar{\chi}\left(\prod_{i \notin S} n_i\right).$$

Now average over the non-trivial characters χ , using

$$\left\langle \chi\left(\prod_{j \in S} n_j\right) \bar{\chi}\left(\prod_{i \notin S} n_i\right) \right\rangle_q = \begin{cases} 1, & \prod_{j \in S} n_j \equiv \prod_{i \notin S} n_i \not\equiv 0 \pmod{q}, \\ 0, & \text{any one of } n_j \equiv 0 \pmod{q}, \\ -1/(q-2), & \text{otherwise.} \end{cases}$$

This gives

$$\left\langle (W_f^{\text{osc}}(\chi))^m \right\rangle_q = \sum_{S \subseteq \{1, \dots, m\}} \langle J(S) \rangle_q$$

with

$$\begin{aligned} \langle J(S) \rangle_q &= \left(-\frac{1}{\log q}\right)^m \sum_{\prod_{j \in S} n_j \equiv \prod_{i \notin S} n_i \pmod{q}} \prod_{j=1}^m \frac{\Lambda(n_j)}{\sqrt{n_j}} \widehat{f}\left(\frac{\log n_j}{\log q}\right) \\ &\quad + \mathcal{O}\left(\frac{1}{q} \left(\frac{1}{\log q} \sum_n \frac{\Lambda(n)}{\sqrt{n}} \widehat{f}\left(\frac{\log n}{\log q}\right)\right)^m\right). \end{aligned} \tag{5.2}$$

To bound the remainder term, use $\text{supp } \widehat{f} \subseteq [-\alpha, \alpha]$ with $\alpha < 2/m$ to estimate the sum

$$\sum_n \frac{\Lambda(n)}{\sqrt{n}} \widehat{f}\left(\frac{\log n}{\log q}\right) \ll \sum_{n \ll q^\alpha} \frac{\Lambda(n)}{\sqrt{n}} \ll q^{\alpha/2}.$$

Thus the \mathcal{O} -term in (5.2) is bounded by

$$\frac{1}{q(\log q)^m} q^{m\alpha/2} \ll q^{-(1-m\alpha/2)} \rightarrow 0.$$

Thus we find

$$\langle J(S) \rangle_q = \left(-\frac{1}{\log q}\right)^m \sum_{\prod_{j \in S} n_j \equiv \prod_{i \notin S} n_i \pmod q} \prod_{j=1}^m \frac{\Lambda(n_j)}{\sqrt{n_j}} \widehat{f}\left(\frac{\log n_j}{\log q}\right) + \mathcal{O}(q^{-(1-m\alpha/2)}).$$

We split the sum for $\langle J(S) \rangle_q$ into terms $J_{\text{eq}}(S)$ where we have equality $\prod_{j \in S} n_j = \prod_{i \notin S} n_i$ rather than mere congruence modulo q , and the remaining terms:

$$\langle J(S) \rangle_q = J_{\text{eq}}(S) + J_{\text{cong}}(S) + \mathcal{O}(q^{-(1-m\alpha/2)})$$

with

$$\begin{aligned} J_{\text{eq}}(S) &:= \left(-\frac{1}{\log q}\right)^m \sum_{\prod_{j \in S} n_j = \prod_{i \notin S} n_i} \prod_{j=1}^m \frac{\Lambda(n_j)}{\sqrt{n_j}} \widehat{f}\left(\frac{\log n_j}{\log q}\right), \\ J_{\text{cong}}(S) &:= \left(-\frac{1}{\log q}\right)^m \sum_{\substack{\prod_{j \in S} n_j \equiv \prod_{i \notin S} n_i \pmod q \\ \prod_{j \in S} n_j \neq \prod_{i \notin S} n_i}} \prod_{j=1}^m \frac{\Lambda(n_j)}{\sqrt{n_j}} \widehat{f}\left(\frac{\log n_j}{\log q}\right). \end{aligned} \tag{5.3}$$

5.1. *Eliminating congruential terms*

We will show that the terms $J_{\text{cong}}(S)$ are negligible:

$$J_{\text{cong}}(S) \ll q^{-(1-m\alpha/2)+\epsilon}$$

for all $\epsilon > 0$.

LEMMA 5.2 *Assume $XY = o(q^2)$. Then*

$$\sum_{\substack{M < X, N < Y \\ M \neq N \\ M \equiv N \pmod q}} \frac{1}{\sqrt{MN}} \ll \frac{\sqrt{XY}}{q}$$

and moreover if $\max(X, Y) = o(q)$ then the sum is empty, hence is equal to zero.

Proof. We may assume $X \leq Y$ and so certainly $X = o(q)$. Then in the sum we must have $M < N$ since otherwise $M = kq + N$ with $k \geq 1$ and so $q < M < X = o(q)$ which gives a contradiction. If $Y = o(q)$ then likewise the sum is empty. Thus we now assume that $Y \gg q$. In that case we write $N = kq + M$, $1 \leq k \leq Y/q$. Then our sum is

$$\begin{aligned} \sum_{M < X} \frac{1}{\sqrt{M}} \sum_{k \leq Y/q} \frac{1}{\sqrt{kq + M}} &\ll \sum_{M < X} \frac{1}{\sqrt{M}} \sum_{k \leq Y/q} \frac{1}{\sqrt{kq}} \\ &\ll \sum_{M < X} \frac{1}{\sqrt{M}} \frac{\sqrt{Y}}{q} \ll \frac{\sqrt{XY}}{q} \end{aligned}$$

as required.

LEMMA 5.3 *Assume that $r + s = m$, and that $\alpha < 2/m$. Then*

$$\frac{1}{(\log q)^m} \sum_{\substack{n_1, \dots, n_r, m_1, \dots, m_s < q^\alpha \\ \prod n_i \equiv \prod m_j \pmod{q} \\ \prod n_i \neq \prod m_j}} \prod \frac{\Lambda(n_i)}{\sqrt{n_i}} \prod \frac{\Lambda(m_j)}{\sqrt{m_j}} \ll q^{m\alpha/2-1+\epsilon}.$$

Proof. We replace the sum over prime powers by the sum over all integers and since all variables are bounded by q^α , we replace $\Lambda(n)$ by $\log q$. Thus our sum is much less than

$$\sum_{\substack{n_1, \dots, n_r, m_1, \dots, m_s < q^\alpha \\ \prod n_i \equiv \prod m_j \pmod{q} \\ \prod n_i \neq \prod m_j}} \prod \frac{1}{\sqrt{n_i}} \prod \frac{1}{\sqrt{m_j}}.$$

Now set $N = \prod n_i$, $M = \prod m_j$ and sum separately over those tuples n_1, \dots, n_r with N, M fixed. The number of such tuples is $\ll q^\epsilon$ for all $\epsilon > 0$. Thus our sum is much less than

$$q^\epsilon \sum_{\substack{M < q^{s\alpha}, N < q^{r\alpha} \\ M \equiv N \pmod{q} \\ M \neq N}} \frac{1}{\sqrt{MN}}$$

which by Lemma 5.2 is $\ll q^{\epsilon+m\alpha/2-1}$ for all $\epsilon > 0$, since $q^{r\alpha} \cdot q^{s\alpha} = o(q^2)$ if $m\alpha = (r+s)\alpha < 2$.

Thus we find that

$$\left\langle (W_f^{\text{osc}})^m \right\rangle_q = \sum_{S \subset \{1, \dots, m\}} J_{\text{eq}}(S) + \mathcal{O}(q^{-(1-m\alpha/2)+\epsilon}) \quad (5.4)$$

for all $\epsilon > 0$.

5.2. Reduction to diagonal terms

Fix a subset $S \subseteq \{1, \dots, m\}$. The sum in $J_{\text{eq}}(S)$ (5.3) is over tuples (n_1, \dots, n_m) which satisfy $\prod_{j \in S} n_j = \prod_{i \notin S} n_i$. We say that there is a *perfect matching* of terms if there is a bijection σ of S onto its complement S^c in $\{1, \dots, m\}$ so that $n_j = n_{\sigma(j)}$ for all $j \in S$. This can happen only if $m = 2k$ is even and $\#S = \#S^c = k$.

Decompose

$$J_{\text{eq}}(S) = J_{\text{diag}}(S) + J_{\text{non}}(S) \quad (5.5)$$

where $J_{\text{diag}}(S)$ is the sum of matching terms—the diagonal part of the sum (non-existent for most S), and $J_{\text{non}}(S)$ is the sum over the remaining, non-matching, terms.

5.3. Diagonal terms

Assume that $m = 2k$ is even. The diagonal terms are the sum over all $\binom{2k}{k}$ subsets $S \subset \{1, \dots, 2k\}$ of cardinality $k = m/2$ and for each such subset S , $J_{\text{diag}}(S)$ is the sum over all $k!$ bijections $\sigma : S \rightarrow S^c$ of S onto its complement, of terms

$$\left(\frac{1}{(\log q)^2} \sum_n \frac{\Lambda(n)^2}{n} \widehat{f} \left(\frac{\log n}{\log q} \right)^2 \right)^k.$$

We evaluate each factor by using the prime number theorem:

$$\begin{aligned} \frac{1}{(\log q)^2} \sum_n \frac{\Lambda(n)^2}{n} \widehat{f}\left(\frac{\log n}{\log q}\right)^2 &\sim \frac{1}{(\log q)^2} \int_2^\infty \frac{\log t}{t} \widehat{f}\left(\frac{\log t}{\log q}\right)^2 dt \\ &\sim \int_0^\infty u \widehat{f}(u)^2 du. \end{aligned} \tag{5.6}$$

Since our function is even and supported inside $(-2/m, 2/m) \subseteq (-1, 1)$ (since $m \geq 2$), we can rewrite this as

$$\frac{1}{2} \int_{-\infty}^\infty \min(1, |u|) \widehat{f}(u)^2 du =: \sigma(f)^2/2.$$

This shows that for $m = 2k$ even we have as $q \rightarrow \infty$ that

$$\sum_{S \subset \{1, \dots, m\}} J_{\text{diag}}(S) \rightarrow \frac{(2k)!}{2^k k!} \sigma(f)^{2k}.$$

Below we will show that the non-diagonal terms $J_{\text{non}}(S)$ are negligible, and hence by (5.4) and (5.5) we will have thus proved Theorem 5.1.

5.4. *Bounding the off-diagonal terms $J_{\text{non}}(S)$*

We will show the following.

LEMMA 5.4

$$J_{\text{non}}(S) \ll \frac{1}{\log q}.$$

Proof. Since

$$\frac{1}{\log q} \sum_p \sum_{k \geq 3} \frac{\log p}{p^{k/2}} \ll \frac{1}{\log q} \sum_p \frac{\log p}{p^{3/2}} \ll \frac{1}{\log q}$$

the contribution of cubes and higher prime powers to (5.3) is negligible, and we may assume in $J_{\text{non}}(S)$ that the n_i are either prime or squares of primes (upto a remainder of $\mathcal{O}(1/\log q)$). By the fundamental theorem of arithmetic, an equality $\prod_{i \in S} n_i = \prod_{j \in S^c} n_j$ forces some of the terms to match, and unless there is a perfect matching of all terms, the remaining integers satisfy equalities of the form $n_1 n_2 = n_3$ with $n_1 = n_2 = p$ prime and $n_3 = p^2$ a square of that prime. Thus upto a remainder of $\mathcal{O}(1/\log q)$, $J_{\text{non}}(S)$ is a sum of terms of the form

$$\left(\frac{1}{(\log q)^2} \sum_{k=1,2} \frac{(\log p)^2}{p^k} \widehat{f}\left(\frac{\log p^k}{\log q}\right)^2 \right)^u \cdot \left(\frac{1}{(\log q)^3} \sum_p \frac{(\log p)^3}{p^2} \widehat{f}\left(\frac{\log p}{\log q}\right)^2 \widehat{f}\left(\frac{\log p^2}{\log q}\right) \right)^v$$

with $2u + 3v = m$, and $v \geq 1$.

We showed (5.6) that the matching terms have an asymptotic value, hence are bounded. We bound the second type of term by

$$\frac{1}{(\log q)^3} \sum_p \frac{(\log p)^3}{p^2} \widehat{f}\left(\frac{\log p}{\log q}\right)^2 \widehat{f}\left(\frac{\log p^2}{\log q}\right) \ll \frac{1}{(\log q)^3} \sum_p \frac{(\log p)^3}{p^2} \ll \frac{1}{(\log q)^3}.$$

Thus as long as $v \geq 1$ (that is, if there is no perfect matching of all terms), we get that the contribution of $J_{\text{non}}(S)$ is $\mathcal{O}(1/\log q)$. This proves Lemma 5.4.

6. The random matrix model

Let $f(x)$ be an even real function subject to the decay condition that there exists a fixed $\epsilon > 0$ and $A > 0$ such that

$$f(x) < A(1 + |x|)^{-(1+\epsilon)} \text{ for all } x \in \mathbb{R}. \quad (6.1)$$

Define

$$F_N(\theta) := \sum_{j=-\infty}^{\infty} f\left(\frac{N}{2\pi}(\theta + 2\pi j)\right)$$

so that $F_N(\theta)$ is 2π -periodic. Define

$$Z_f(U) := \sum_{j=1}^N F_N(\theta_j),$$

where U is an $N \times N$ unitary matrix with eigenangles $\theta_1, \dots, \theta_N$. This is the random matrix equivalent of $W_f(\chi)$.

The Fourier coefficients of $F_N(\theta)$ are

$$\begin{aligned} a_{n,N} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} F_N(\theta) e^{-in\theta} d\theta \\ &= \frac{1}{N} \int_{-\infty}^{\infty} f(x) e^{-2\pi inx/N} dx = \frac{1}{N} \widehat{f}\left(\frac{n}{N}\right) \end{aligned}$$

and so, if the matrix U has eigenangles $\theta_1, \dots, \theta_N$,

$$Z_f(U) := \sum_{j=1}^N F_N(\theta_j) = \sum_{n=-\infty}^{\infty} \frac{1}{N} \widehat{f}\left(\frac{n}{N}\right) \text{Tr } U^n. \quad (6.2)$$

Since

$$\mathbb{E} \{ \text{Tr } U^n \} = \begin{cases} N, & n = 0, \\ 0, & \text{otherwise} \end{cases}$$

we have thus proven the following.

THEOREM 6.1

$$\mathbb{E} \{ Z_f \} = \widehat{f}(0).$$

The definition of Fourier transform we use is such that $\widehat{f}(0) = \int_{-\infty}^{\infty} f(x) dx$, so this theorem is in perfect agreement with Theorem 3.1.

THEOREM 6.2 *The variance of Z_f tends to σ^2 as $N \rightarrow \infty$, where*

$$\sigma^2 = \int_{-\infty}^{\infty} \min(|u|, 1) \widehat{f}(u)^2 du. \quad (6.3)$$

Proof. Since [4, 6, 16]

$$\mathbb{E} \{ \text{Tr } U^n \text{Tr } U^m \} = \begin{cases} N^2 & \text{if } n = m = 0, \\ |n| & \text{if } n = -m \text{ and } |n| \leq N, \\ N & \text{if } n = -m \text{ and } |n| \geq N, \\ 0 & \text{otherwise} \end{cases}$$

we have

$$\begin{aligned} \mathbb{E} \{ (Z_f - \widehat{f}(0))^2 \} &= \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{1}{N^2} \widehat{f}\left(\frac{n}{N}\right) \widehat{f}\left(-\frac{n}{N}\right) \min(|n|, N) \\ &\rightarrow \int_{-\infty}^{\infty} \min(|u|, 1) \widehat{f}(u)^2 du, \end{aligned}$$

the last line following from the definition of a Riemann integral, and from the fact that $f(x)$ is even.

Note that this is the same as the variance of $W_f(\chi)$ (Theorem 4.1) when \widehat{f} is restricted to have support contained in $[-1, 1]$.

6.1. *Mock-Gaussian behaviour*

From (6.2) and Theorem 6.1, the m th centred moment is

$$\mathbb{E} \{ (Z_f - \mathbb{E}\{Z_f\})^m \} = \sum_{\substack{n_1=-\infty \\ n_1 \neq 0}}^{\infty} \cdots \sum_{\substack{n_m=-\infty \\ n_m \neq 0}}^{\infty} \frac{1}{N} \widehat{f}\left(\frac{n_1}{N}\right) \cdots \frac{1}{N} \widehat{f}\left(\frac{n_m}{N}\right) \mathbb{E} \{ \text{Tr } U^{n_1} \cdots \text{Tr } U^{n_m} \}.$$

The following two lemmas will enable us to calculate these moments, under certain restrictions on the support of $\widehat{f}(u)$.

LEMMA 6.3 *If $\sum_{j=1}^m n_j \neq 0$ then*

$$\mathbb{E} \left\{ \prod_{j=1}^m \text{Tr } U^{n_j} \right\} = 0.$$

Proof. By rotation invariance of Haar measure, the left-hand side is left unchanged by multiplication by $Ie^{i\theta}$ (where θ is an arbitrary angle, and I is the identity matrix). Since $\text{Tr}((UIe^{i\theta})^n) = e^{in\theta} \text{Tr } U^n$, this means

$$\mathbb{E} \left\{ \prod_{j=1}^m \text{Tr } U^{n_j} \right\} = \exp\left(i\theta \sum_{j=1}^m n_j\right) \mathbb{E} \left\{ \prod_{j=1}^m \text{Tr } U^{n_j} \right\}$$

which is true only if either both sides are zero, or if $\sum_{j=1}^m n_j = 0$.

LEMMA 6.4 (Diaconis, Shahshahani) [4, 5] For $a_j, b_j \in \{0, 1, 2, \dots\}$, if

$$N \geq \max \left(\sum_{j=1}^k j a_j, \sum_{j=1}^k j b_j \right),$$

then

$$\mathbb{E} \left\{ \prod_{j=1}^k (\text{Tr } U^j)^{a_j} (\text{Tr } U^{-j})^{b_j} \right\} = \delta_{a,b} \prod_{j=1}^k j^{a_j} a_j!,$$

where $\delta_{a,b} = 1$ if $a_j = b_j$ for $j = 1, \dots, k$, and $\delta_{a,b} = 0$ otherwise.

THEOREM 6.5 For any integer $m \geq 2$, if $\text{supp } \widehat{f}(u) \subseteq [-2/m, 2/m]$, then

$$\lim_{N \rightarrow \infty} \mathbb{E} \left\{ (Z_f - \widehat{f}(0))^m \right\} = \begin{cases} 0 & \text{if } m \text{ odd,} \\ \frac{(2k)!}{2^k k!} \sigma^m & \text{if } m = 2k, k \geq 1 \text{ an integer,} \end{cases}$$

where σ^2 , the variance, is given by (6.3).

Proof. The restriction on the support of $\widehat{f}(u)$ gives

$$\begin{aligned} & \mathbb{E} \left\{ (Z_f - \widehat{f}(0))^m \right\} \\ &= \frac{1}{N^m} \sum_{\substack{n_1 = -2N/m \\ n_1 \neq 0}}^{2N/m} \cdots \sum_{\substack{n_m = -2N/m \\ n_m \neq 0}}^{2N/m} \widehat{f} \left(\frac{n_1}{N} \right) \cdots \widehat{f} \left(\frac{n_m}{N} \right) \mathbb{E} \left\{ \text{Tr } U^{n_1} \cdots \text{Tr } U^{n_m} \right\}. \end{aligned} \quad (6.4)$$

Lemma 6.3 means that to have a non-zero contribution, $\sum n_j = 0$, and so

$$\max_{\substack{n_j \\ |n_j| < 2N/m \\ \sum n_j = 0}} \left\{ \sum_{j=1}^m n_j \mathbb{1}_{\{n_j > 0\}} \right\} \leq \frac{m}{2} \frac{2N}{m} = N,$$

the maximum is obtained by all the positive terms equal to $2N/m$, and all the negative terms to $-2N/m$. (This maximum is obtainable only if m is even.) Thus we see that the support restriction means all the non-zero terms in (6.4) can be calculated using Lemma 6.4.

To obtain anything non-zero using Lemma 6.4, there must be a bijection σ mapping $\{1, \dots, m\}$ into itself so that $n_j = -n_{\sigma(j)}$ for all j . Note that no n_j can equal zero, since this is expressly forbidden in (6.4).

For odd m , it is impossible to pair off the n_j without having at least one $n_j = 0$. Therefore (6.4) is zero for m odd.

For even $m = 2k$, assume the n_j are such that they can be paired off, and relabel so that $r_1 = n_{j_1}$ where j_1 is the smallest number such that $n_{j_1} > 0$, $r_2 = n_{j_2}$ where j_2 is the second smallest number such that $n_{j_2} > 0$, etc. There are $\binom{2k}{k}$ ways of arranging the positive $n_j > 0$ to give the same r_i .

The number of ways of ordering the negative n_j such that each positive term has a negative partner equals $\frac{k!}{b_1! b_2! \dots}$ where $b_i = \#\{j : n_j = -i\}$. Therefore, after reordering, (6.4) equals

$$\begin{aligned} & \sum_{r_1=1}^{N/k} \dots \sum_{r_k=1}^{N/k} \binom{2k}{k} \frac{k!}{b_1! b_2! \dots} \mathbb{E} \left\{ |\text{Tr } U^{r_1}|^2 \dots |\text{Tr } U^{r_k}|^2 \right\} \prod_{i=1}^k \frac{1}{N^2} \widehat{f}\left(\frac{r_i}{N}\right) \widehat{f}\left(\frac{-r_i}{N}\right) \\ &= \frac{(2k)!}{k!} \left(\sum_{r=1}^{N/k} r \left| \frac{1}{N} \widehat{f}\left(\frac{r}{N}\right) \right|^2 \right)^k \\ &\rightarrow \frac{(2k)!}{k! 2^k} \sigma^2 \end{aligned}$$

since

$$\frac{1}{b_1! b_2! \dots} \mathbb{E} \left\{ |\text{Tr } U^{r_1}|^2 \dots |\text{Tr } U^{r_k}|^2 \right\} = \prod_{j=1}^k r_j$$

by Lemma 6.4, and since

$$\sum_{r=1}^{N/k} r \left| \frac{1}{N} \widehat{f}\left(\frac{r}{N}\right) \right|^2 \sim \frac{1}{2} \int_{-1/k}^{1/k} |u| |\widehat{f}(u)|^2 du = \frac{1}{2} \sigma^2$$

when $\text{supp } \widehat{f} \subseteq [-1/k, 1/k]$, where σ^2 is given by (6.3).

REMARK 3 One can also prove Theorem 6.5 by a completely different method, using techniques found in [21]; we need to take this route when dealing with the other classical groups in [8].

7. Unrestricted moments of $Z_f(U)$

In this section we will calculate the (uncentred) m th moment of $Z_f(U)$ without restriction on the support. This allows us to conjecture an extension to Theorems 3.1 and 5.1. In particular, it appears that the m th centred moment of $W_f(\chi)$ is not Gaussian outside the range given in Theorem 5.1, which would imply that $W_f(\chi)$ does not converge to a normal distribution.

We wish to calculate the (uncentred) m th moment of $Z_f(U)$:

$$\begin{aligned} M_m &:= \lim_{N \rightarrow \infty} \mathbb{E} \left\{ (Z_f)^m \right\} \\ &= \lim_{N \rightarrow \infty} \mathbb{E} \left\{ \sum_{i_1=1}^N \dots \sum_{i_n=1}^N F_N(\theta_{i_1}) \dots F_N(\theta_{i_n}) \right\}. \end{aligned} \tag{7.1}$$

To evaluate M_m , we use the r -point correlation function of Dyson.

LEMMA 7.1 (Dyson) For an arbitrary function g of r variables which is 2π -periodic in all its variables,

$$\mathbb{E} \left\{ \sum_{\substack{i_1, \dots, i_r=1 \\ i_j \text{ all distinct}}}^N g(\theta_{i_1}, \dots, \theta_{i_r}) \right\} = \frac{1}{(2\pi)^r} \int_{-\pi}^{\pi} \dots \int_{-\pi}^{\pi} g(\theta_1, \dots, \theta_r) R_r^{(N)}(\theta_1, \dots, \theta_r) d\theta_1 \dots d\theta_r,$$

where

$$R_r^{(N)}(\theta_1, \dots, \theta_r) = \det \{ S_N(\theta_j - \theta_i) \}_{1 \leq i, j \leq r}$$

with

$$S_N(x) = \frac{\sin(Nx/2)}{\sin(x/2)}.$$

Note that the sums in (7.1) range unrestrictedly over all variables (they include both diagonal and off-diagonal terms), whereas Lemma 7.1 requires the sums to be over distinct variables (off-diagonals only). We overcome this problem by summing over the diagonals separately.

DEFINITION 7.2 σ is said to be a set partition of m elements into r non-empty blocks if

$$\sigma : \{1, \dots, m\} \longrightarrow \{1, \dots, r\}$$

satisfying the following.

- (1) For every $q \in \{1, \dots, r\}$ there exists at least one j such that $\sigma(j) = q$ (this is the non-emptiness of the blocks).
- (2) For all j , either $\sigma(j) = 1$ or there exists a $k < j$ such that $\sigma(j) = \sigma(k) + 1$. (Roughly speaking, if we think of $\{1, \dots, r\}$ as denoting ordered pigeonholes, then $\sigma(j)$ either goes into a non-empty pigeonhole, or into the next empty hole).

The collection of all set partitions of m elements into r blocks is denoted by $P(m, r)$.

REMARK 4 The number of $\sigma \in P(m, r)$ is equal to $S(m, r)$, a Stirling number of the second kind. The number of set partitions of m elements into any number of non-empty blocks is $\sum_{r=1}^m S(m, r) = B_m$, a Bell number.

LEMMA 7.3 For any function g of m variables,

$$\sum_{j_1, \dots, j_m} g(x_{j_1}, \dots, x_{j_m}) = \sum_{r=1}^m \sum_{\sigma \in P(m, r)} \sum_{\substack{i_1, \dots, i_r \\ i_j \text{ all distinct}}} g(x_{i_{\sigma(1)}}, \dots, x_{i_{\sigma(m)}}).$$

Proof. Each term on the left appears once and only once on the right, so they are equal.

THEOREM 7.4

$$M_m = \sum_{r=1}^m \underbrace{\int \dots \int}_{- \infty}^{\infty} R_r(x_1, \dots, x_r) \sum_{\sigma \in P(m, r)} \prod_{q=1}^r f^{\lambda_q}(x_q) dx_q,$$

where $\lambda_q = \#\{j : \sigma(j) = q\}$, and where

$$R_r(x_1, \dots, x_r) = \det \left\{ \frac{\sin(\pi(x_j - x_i))}{\pi(x_j - x_i)} \right\}_{1 \leq i, j \leq r}.$$

Proof. Recall (7.1) that

$$M_m = \lim_{N \rightarrow \infty} \mathbb{E} \left\{ \sum_{i_1=1}^N \cdots \sum_{i_m=1}^N F_N(\theta_{i_1}) \cdots F_N(\theta_{i_m}) \right\}.$$

Lemma 7.3 gives

$$\begin{aligned} & \mathbb{E} \left\{ \sum_{i_1=1}^N \cdots \sum_{i_m=1}^N F_N(\theta_{i_1}) \cdots F_N(\theta_{i_m}) \right\} \\ &= \sum_{r=1}^m \sum_{\sigma \in P(m,r)} \mathbb{E} \left\{ \sum_{\substack{i_1, \dots, i_r=1 \\ i_j \text{ all distinct}}}^N F_N(\theta_{i_{\sigma(1)}}) \cdots F_N(\theta_{i_{\sigma(m)}}) \right\} \\ &= \sum_{r=1}^m \sum_{\sigma \in P(m,r)} \mathbb{E} \left\{ \sum_{\substack{i_1, \dots, i_r=1 \\ i_j \text{ all distinct}}}^N F_N^{\lambda_1}(\theta_{i_1}) \cdots F_N^{\lambda_r}(\theta_{i_r}) \right\}, \quad (7.2) \end{aligned}$$

where $\lambda_q = \#\{j : \sigma(j) = q\}$. Lemma 7.1 now applies, and gives

$$\begin{aligned} & \mathbb{E} \left\{ \sum_{\substack{i_1, \dots, i_r=1 \\ i_j \text{ all distinct}}}^N F_N^{\lambda_1}(\theta_{i_1}) \cdots F_N^{\lambda_r}(\theta_{i_r}) \right\} \\ &= \frac{1}{(2\pi)^r} \int \cdots \int_{-\pi}^{\pi} F_N^{\lambda_1}(\theta_1) \cdots F_N^{\lambda_r}(\theta_r) R_r^{(N)}(\theta_1, \dots, \theta_r) d\theta_1 \cdots d\theta_r \\ &= \frac{1}{N^r} \int \cdots \int_{-N/2}^{N/2} R_r^{(N)}\left(\frac{2\pi x_1}{N}, \dots, \frac{2\pi x_r}{N}\right) \prod_{q=1}^r F_N^{\lambda_q}\left(\frac{2\pi x_q}{N}\right) dx_q \end{aligned}$$

upon change of variables to $x_n = \frac{N}{2\pi}\theta_n$. Now,

$$F_N\left(\frac{2\pi x}{N}\right) = \sum_{j=-\infty}^{\infty} f(x + Nj) = f(x) + \mathcal{O}\left(\frac{1}{N^{1+\epsilon}}\right)$$

uniformly for all $x \in [-N/2, N/2]$, due to the decay condition on f (6.1).

Since

$$\lim_{N \rightarrow \infty} \frac{1}{N^r} R_r^{(N)}\left(\frac{2\pi x_1}{N}, \dots, \frac{2\pi x_r}{N}\right) = R_r(x_1, \dots, x_r),$$

where

$$R_r(x_1, \dots, x_r) = \det \left\{ \frac{\sin(\pi(x_j - x_i))}{\pi(x_j - x_i)} \right\}_{1 \leq i, j \leq r},$$

we have

$$\lim_{N \rightarrow \infty} \mathbb{E} \left\{ \sum_{\substack{i_1, \dots, i_r=1 \\ i_j \text{ all distinct}}}^N F_N^{\lambda_1}(\theta_{i_1}) \dots F_N^{\lambda_r}(\theta_{i_r}) \right\} = \int \dots \int_{-\infty}^{\infty} R_r(x_1, \dots, x_r) \prod_{q=1}^r f^{\lambda_q}(x_q) dx_q. \quad (7.3)$$

Hence, combining (7.1), (7.2) and (7.3)

$$\begin{aligned} M_m &= \lim_{N \rightarrow \infty} \mathbb{E} \left\{ \sum_{i_1=1}^N \dots \sum_{i_m=1}^N F_N(\theta_{i_1}) \dots F_N(\theta_{i_m}) \right\} \\ &= \sum_{r=1}^m \sum_{\sigma \in P(m,r)} \underbrace{\int \dots \int_{-\infty}^{\infty}}_r R_r(x_1, \dots, x_r) \prod_{q=1}^r f^{\lambda_q}(x_q) dx_q \end{aligned} \quad (7.4)$$

as required.

REMARK 5 One can show that the moments M_m of Z_f uniquely determine the distribution to which Z_f weakly converges as $N \rightarrow \infty$.

8. Application: small first zeros of $L(s, \chi)$

In this section we will apply the results of sections 3 and 4 to show that, under the assumption of GRH there exist Dirichlet L -functions with first zero lower than the expected height. Small gaps between high zeros of the Riemann zeta function (which also obey unitary statistics) have been much studied. Montgomery [11] showed that an infinite number of zeros are less than 0.68 times their average spacing. This was reduced to 0.5179 by Montgomery and Odlyzko [12]; to 0.5171 by Conrey, Ghosh and Gonek [2]; and to 0.5169 by Conrey and Iwaniec, as announced in [1]. Conrey, Ghosh, Goldston, Gonek and Heath-Brown [3] showed that a positive proportion of zeros are less than 0.77 times the average spacing, a result improved to 0.6878 by Soundararajan [20]. We should perhaps point out that the main difference between gaps between the zeta zeros, and the height of the lowest Dirichlet zero is that the point $\frac{1}{2}$ is not expected to ‘repel’ low-lying zeros.

8.1. *Infinitely many small first zeros*

Using Theorem 3.1 we are able to obtain some partial results for extreme low-lying zeros of Dirichlet L -functions.

THEOREM 8.1 *Assume GRH. If*

$$\lim_{q \rightarrow \infty} \langle W_f \rangle_q = \int_{-\infty}^{\infty} f(x) dx \quad (8.1)$$

for all admissible functions f with $\text{supp } \widehat{f} \subseteq [-2R, 2R]$, then

$$\liminf_{q \rightarrow \infty} \min_{\chi \neq \chi_0} x_{\chi,1} \leq \frac{1}{4R},$$

where the minimum of the first zero of $L(s, \chi)$ is taken over all non-trivial characters modulo q .

Proof. Let $\widehat{g}(u)$ be an even, continuous function, with $\text{supp } \widehat{g} \subset [-R, R]$, and such that $\widehat{g}(u)$ is differentiable in $[-R, R]$ and $g(x) \ll |x|^{-3/2-\delta}$, $\delta > 0$.

Let

$$B := \sqrt{\frac{\int_0^\infty x^2 g^2(x) dx}{\int_0^\infty g^2(x) dx}} = \sqrt{\frac{(1/4\pi^2) \int_0^\infty \widehat{g}'(u)^2 du}{\int_0^\infty \widehat{g}(u)^2 du}}$$

so, by assumptions on $\widehat{g}(u)$ and its derivative, we see that B is a strictly positive finite real number.

Define, for $\beta > B$,

$$f(x) = (x^2 - \beta^2)g^2(x)$$

so that f has the properties

$$\int_0^\infty f(x) dx = -(\beta^2 - B^2) \int_0^\infty g^2(x) dx < 0 \tag{8.2}$$

and

$$f(x) \leq 0 \text{ for } |x| \leq \beta.$$

Note that the conditions on $\widehat{g}(u)$ mean that f is an admissible function.

Observe that

$$\widehat{f}(u) = \frac{-1}{4\pi^2} \frac{d^2}{du^2} (\widehat{g} \star \widehat{g})(u) - \beta^2 (\widehat{g} \star \widehat{g})(u) = \frac{-1}{4\pi^2} (\widehat{g}' \star \widehat{g}')(u) - \beta^2 (\widehat{g} \star \widehat{g})(u),$$

where $(\widehat{g} \star \widehat{g})(u)$ is the convolution of \widehat{g} with itself. Since differentiation and multiplication by a constant does not increase the support of a function, we may conclude that $\text{supp } \widehat{f} \subset [-2R, 2R]$ since, by assumption, $\text{supp } \widehat{g} \subset [-R, R]$.

Therefore, by assumption (8.1) and by (8.2),

$$\frac{1}{q-2} \sum_{\chi \neq \chi_0} \sum_{j \geq 1} f(x_{\chi,j}) \sim \int_0^\infty f(x) dx < 0.$$

By the assumption of GRH all the $x_{\chi,j}$ are real, and so we may conclude that there exists a q_0 such that for all $q > q_0$,

$$\frac{1}{q-2} \sum_{\chi \neq \chi_0} \sum_{\substack{j \geq 1 \\ x_{\chi,j} \leq \beta}} f(x_{\chi,j}) < \frac{1}{q-2} \sum_{\chi \neq \chi_0} \sum_{j \geq 1} f(x_{\chi,j}) < 0$$

and so, for all $q > q_0$ at least one χ , a non-trivial character modulo q , exists with $x_{\chi,1} \leq \beta$. (Note that this method produces a non-vacuous result only if $\beta < 1$, since by definition, $\langle x_{\chi,1} \rangle_q \rightarrow 1$.) The theorem will follow if we can construct a $g(x)$ satisfying all the conditions such that $B = 1/4R$, by letting $\beta \rightarrow B$.

Taking $\widehat{g}(u) = \cos(\pi u/2R) \mathbb{1}_{\{|u| \leq R\}}$, so that

$$g(x) = \frac{-4R \cos(2\pi x R)}{\pi(16x^2 R^2 - 1)},$$

we see that

$$B^2 = \frac{\int_0^\infty x^2 g^2(x) dx}{\int_0^\infty g^2(x) dx} = \frac{1}{16R^2}.$$

This concludes the proof of Theorem 8.1.

REMARK 6 Our choice of $\widehat{g}(u) = \cos(\pi u/2R) \mathbb{1}_{\{|u| \leq R\}}$ was not an arbitrary one, as this is the optimizing function for this method.

COROLLARY 8.2 *If the generalized Riemann hypothesis holds, then*

$$\liminf_{q \rightarrow \infty} \min_{\chi \neq \chi_0} x_{\chi,1} \leq \frac{1}{4},$$

where the minimum of the first scaled zero of $L(s, \chi)$ is taken over all non-trivial characters modulo q .

Proof. By Theorem 3.1 we may take $R = 1$ in Theorem 8.1.

REMARK 7 Random matrix theory suggests that

$$\liminf_{q \rightarrow \infty} \min_{\chi \neq \chi_0} x_{\chi,1} = 0.$$

8.2. Positive proportion of small first zeros

Theorem 8.1 combined with Theorem 5.1 allows us to deduce a statement about a positive proportion (rather than just infinitely many) of the χ have smaller than expected first zeros.

THEOREM 8.3 *Assume GRH. For $\beta \geq 0.633$,*

$$\liminf_{q \rightarrow \infty} \frac{1}{q-2} \#\{\chi \neq \chi_0 : x_{\chi,1} < \beta\} \geq \frac{11\pi^2 - 3 - 72\beta^2 - 88\pi^2\beta^2 - 48\beta^4 + 176\pi^2\beta^4}{12\pi^2(4\beta^2 - 1)^2}.$$

Proof. Take

$$f_\beta(x) = (x^2 - \beta^2)g^2(x),$$

where

$$\widehat{g}(u) = \cos(\pi u) \mathbb{1}_{\{|u| \leq \frac{1}{2}\}}$$

(so $\widehat{f}(u)$ has support in $[-1, 1]$, and $f(x) \leq 0$ for $|x| \leq \beta$, and $f(x) \geq 0$ otherwise).

As in the proof of Theorem 8.1 we have

$$\lim_{q \rightarrow \infty} \langle W_f \rangle_q = \int_{-\infty}^{\infty} f(x) dx < 0 \text{ for } \beta > \frac{1}{2}. \quad (8.3)$$

By Theorem 5.1,

$$\lim_{q \rightarrow \infty} \left\langle \left(W_f - \langle W_f \rangle_q \right)^2 \right\rangle_q = \int_{-1}^1 |u| |\widehat{f}(u)|^2 du.$$

Chebyshev's inequality gives

$$\limsup_{q \rightarrow \infty} \frac{1}{q-2} \# \left\{ \chi \neq \chi_0 : \left| W_f - \langle W_f \rangle_q \right| \geq \epsilon \right\} \leq \frac{\int_{-1}^1 |u| |\widehat{f}(u)|^2 du}{\epsilon^2}$$

and so, using the fact that f is even,

$$\liminf_{q \rightarrow \infty} \frac{1}{q-2} \# \left\{ \chi \neq \chi_0 : \left| \sum_{j \geq 1} f(x_{\chi,j}) - \int_0^\infty f(x) dx \right| \leq \epsilon_1 \right\} \geq 1 - \frac{\int_{-1}^1 |u| |\widehat{f}(u)|^2 du}{4\epsilon_1^2},$$

where $\epsilon_1 = \epsilon/2$.

If $\beta > \frac{1}{2}$, putting $\epsilon_1 = \left| \int_0^\infty f(x) dx \right| = \frac{1}{2} |\widehat{f}(0)|$, we get

$$\liminf_{q \rightarrow \infty} \frac{1}{q-2} \# \left\{ \chi \neq \chi_0 : -2 \left| \int_0^\infty f(x) dx \right| \leq \sum_{j \geq 1} f(x_{\chi,j}) \leq 0 \right\} \geq 1 - \frac{\int_{-1}^1 |u| |\widehat{f}(u)|^2 du}{\widehat{f}(0)^2}.$$

(Note we need GRH here, so that $\sum_{j \geq 1} f(x_{\chi,j})$ is real.) Since $f(x_{\chi,j}) < 0$ implies $x_{\chi,j} < \beta$ we may conclude that, after working out the integrals on the right-hand side,

$$\liminf_{q \rightarrow \infty} \frac{1}{q-2} \# \left\{ \chi \neq \chi_0 : x_{\chi,1} < \beta \right\} \geq 1 - \frac{3 + \pi^2 + 72\beta^2 - 8\pi^2\beta^2 + 48\beta^4 + 16\pi^2\beta^4}{12\pi^2(4\beta^2 - 1)^2}.$$

The right-hand side is greater than zero for

$$\beta \geq \frac{1}{2} \frac{\sqrt{9 + 11\pi^2 + 2\sqrt{18 + 66\pi^2}}}{\sqrt{11\pi^2 - 3}} \approx 0.633$$

as required.

REMARK 8 Random matrix theory suggests that a positive proportion of the χ have $x_{\chi,1} < \beta$ for any $\beta > 0$.

REMARK 9 The test function we used in the proof is the optimum test function for Theorem 8.1, but that does not necessarily make it the optimum test function here. Indeed, the word ‘optimum’ is not well defined here, as one can either try to find a function that maximizes the estimate of the proportion of χ satisfying $x_{1,\chi} < \beta$, or one could try to find a function which minimizes the β for which this method proves a positive proportion of $x_{1,\chi} \leq \beta$.

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